Refraction at a Spherical Interface*

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Abstract

We look at refraction at a spherical interface in the small angle approximation.

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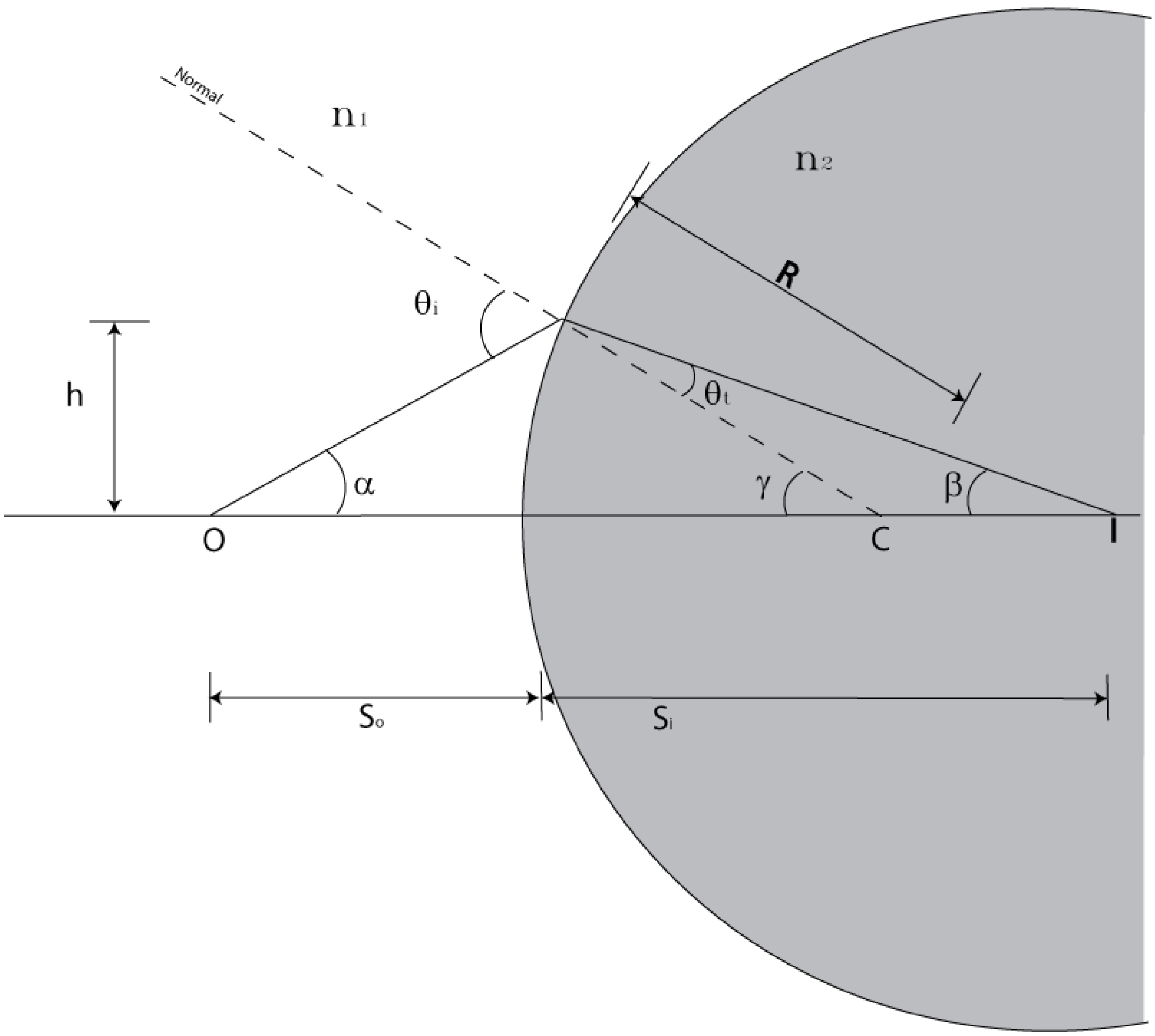

Figure 1: Refraction at a spherical interface. Click on image for larger version.

Look at the figure showing refraction at a sphere.In this figure:

- C is the center of curvature of the spherical surface
- R is the radius of curvature
- O is the position of the Object
- I is the position of the Image
- S_o is the distance of the object from the surface along the optical axis
- S_i is the distance from the surface to the Image
- $n_1 < n2$

There is a ray that strikes the surface at height h. In general rays hitting the surface at different points will be bent to different points along the optical axis. However for small angles we will show they all converge at the same point.So lets use the small angle approximation

$$\tan\alpha = \frac{h}{s_o} \approx \alpha$$

$$\tan\beta = \frac{h}{s_i} \approx \beta$$

$$\tan\gamma = \frac{h}{R} \approx \gamma$$

Now from trigonometry we can see that:

$$\theta_i = \alpha + \gamma$$

$$\gamma = \theta_t + \beta$$

or

$$\theta_t = \gamma - \beta$$

now Snell's law says

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

or

$$n_1 \theta_i \approx n_2 \theta_t$$

$$n_1 (\alpha + \gamma) = n_2 (\gamma - \beta)$$

$$\gamma (n_2 - n_1) = n_2 \beta + n_1 \alpha$$

$$\frac{h}{R} (n_2 - n_1) = n_2 \frac{h}{s_i} + n_1 \frac{h}{s_o}$$

Now all the h's cancel so there is no dependence on point on surface. That is:

$$\frac{n_2 - n_1}{R} = \frac{n_2}{s_i} + \frac{n_1}{s_o}$$

Now lets consider the case of a concave surface. The picture is

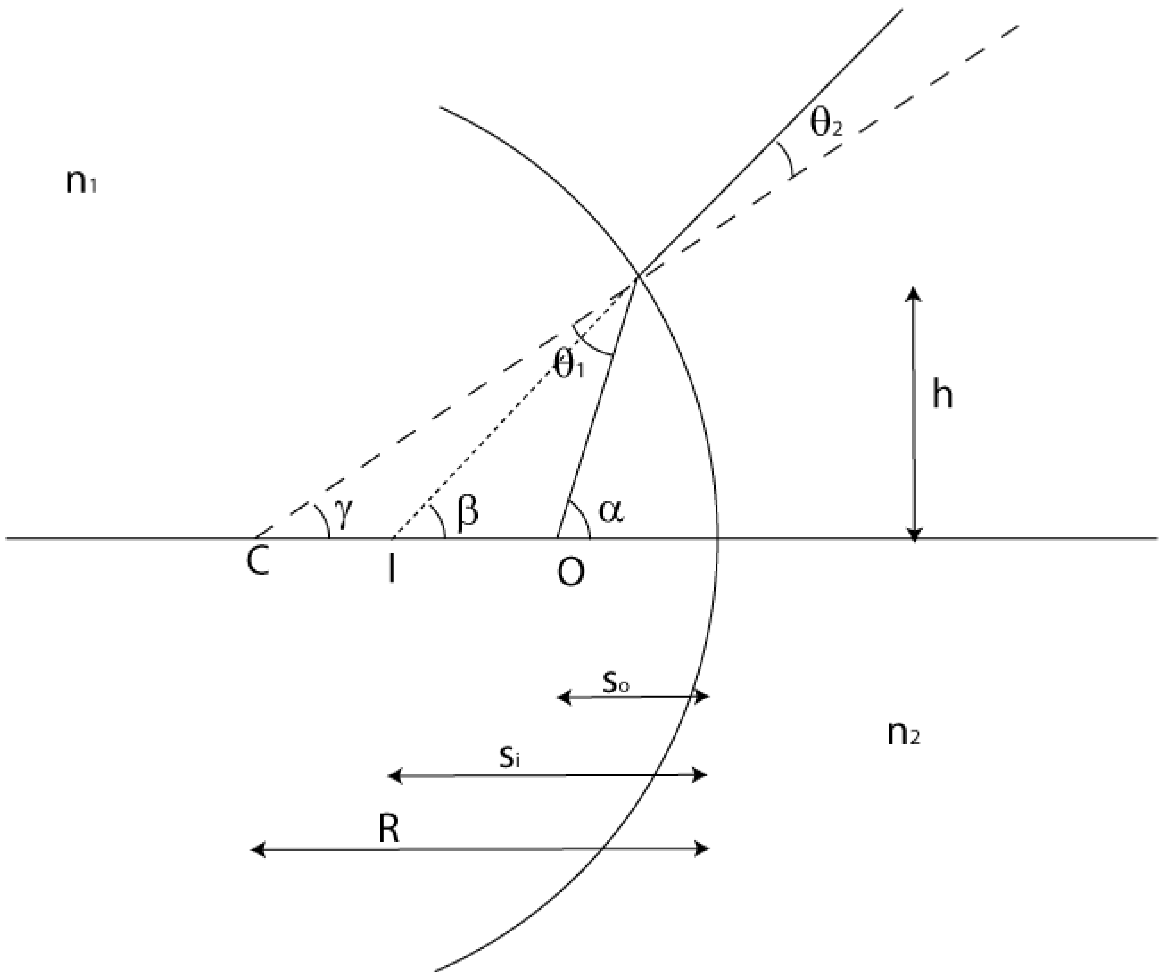

Figure 2: Click to get larger image.

Again we use the small angle approximation and thus we have

$$n_1\theta_1 = n_2\theta_2$$

In this case we also see that

$$\alpha = \theta_1 + \gamma$$

and

$$\beta = \theta_2 + \gamma$$

so we can write

$$n_1\left(\alpha - \gamma\right) = n_2\left(\beta - \gamma\right)$$

or

$$n_1\left(\frac{h}{s_o} - \frac{h}{R}\right) = n_2\left(\frac{h}{s_i} - \frac{h}{R}\right)$$

or

$$\frac{n_1}{s_o} - \frac{n_2}{s_i} = \left(\frac{n_1 - n_2}{R}\right)$$

However we can make the equation identical to the previous one if we adopt the following sign convention:

- s_o is positive to the left of the interface
- s_i is positive to the right of the interface
- R is positive when the center of the sphere is to the right of the interface

Then the equation becomes as before

$$\frac{n_2 - n_1}{R} = \frac{n_2}{s_i} + \frac{n_1}{s_o}$$

In this case note that the image is imaginary (whereas in the first case it was real). Note that the actual rays pass through a real image.

The focal point is the object point which causes the image to occur at infinity.

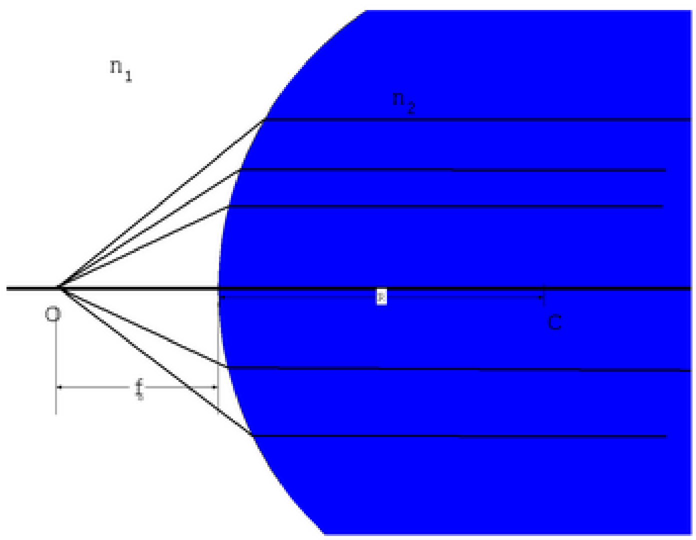

Figure 3

That is all the rays end up traveling parallel to each other. In this case s_i goes to∞ so

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R}$$

becomes

$$\frac{n_1}{f_o} = \frac{n_2 - n_1}{R}$$

or

$$f_o = \frac{n_1 R}{n_2 - n_1}.$$

Now we can find a focal point to the right of the of the surface by considering parallel rays coming in from the left.

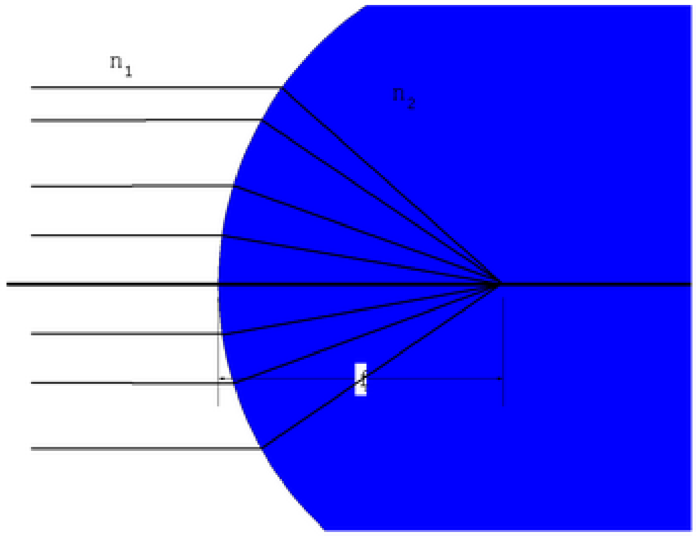

Figure 4

Then we get

$$f_i = \frac{n_2 R}{n_2 - n_1}$$

But we do have to expand our sign conventionfor light incident from the left

- s_o is positive to the left of the interface
- s_i is positive to the right of the interface
- R is positive when the center of the sphere is to the right of the interface
- f_o is positive to the left of the interface
- f_i is positive to the right of the interface

With the definition of focal points, we also have a natural way to graphically solve optical problems. Any ray drawn horizontally from the left side of the interface will pass through the focal point on the right. Any ray going through the focal point on the left will go horizontally on the right. The following figure illustrates this.

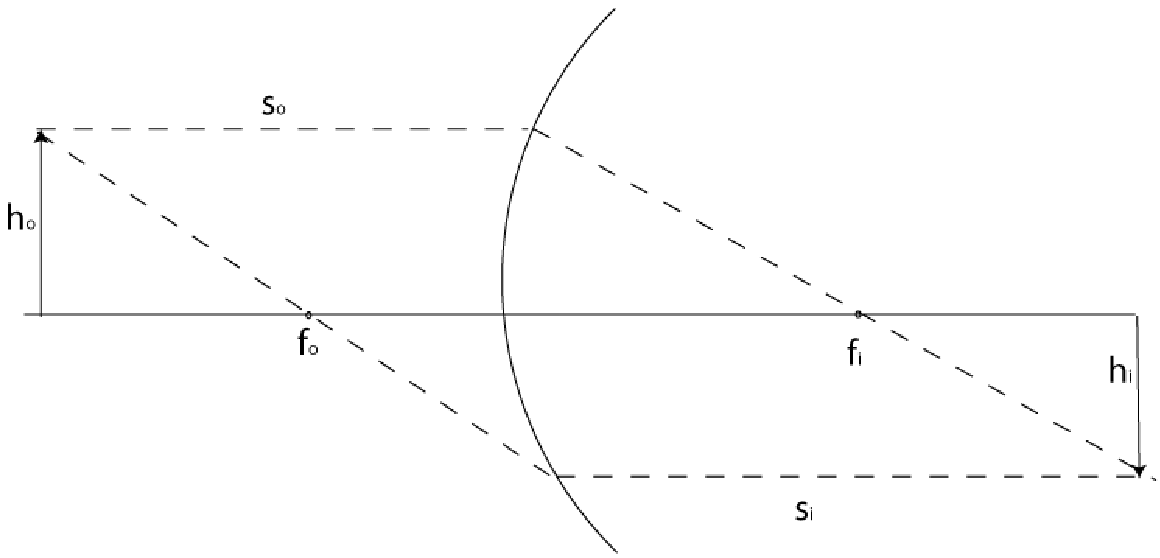

Figure 5

The magnification of the image is the ratio of the heights h_o to h_i.

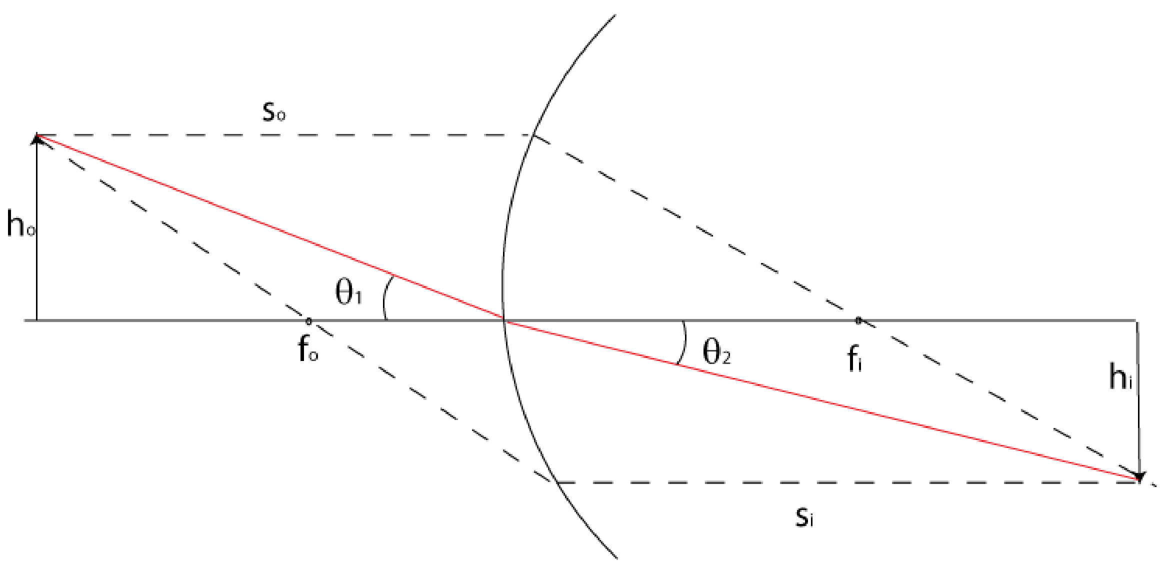

Figure 6

Since we are using the small angle approximation, we have Snell's law

$$n_1\theta_1 = n_2\theta_2$$

which can be rewritten

$$n_1\left(\frac{h_o}{s_o}\right)=n_2\left(\frac{h_i}{s_i}\right).$$

So we write that the magnification is

$$m=\frac{h_i}{h_o}=-\frac{n_1 s_i}{n_2 s_o}.$$

The negative sign is introduced to capture the fact that the image is inverted. It is worth pointing out that in our diagram above, the image is real, because the actual light rays pass through it.