Thick Lenses and Lens Systems(6)

Cardinal Points(6.1)

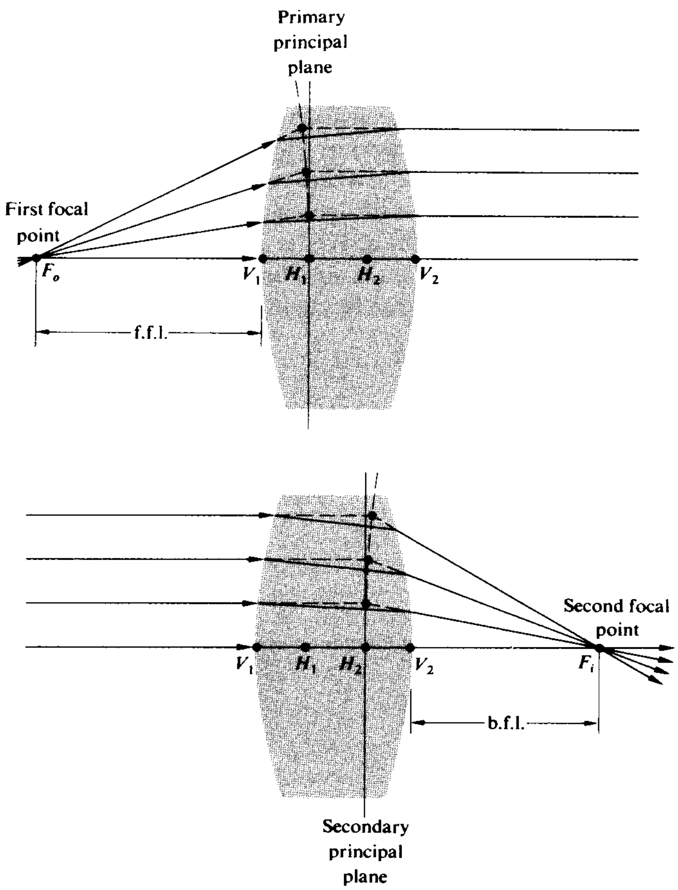


Figure 6.1 A thick lens.

Principle plane: the plane on which the extension lines of the ray incident from the first focus and the ray emerged from the lens intercept.

Secondary Plane: the same as the principle plane except that the ray is from the second focus.

First principal point H_1: the intersection of the **Principle plane** and the optical axis.

Second principal point H_2: the intersection of the **secondary plane** and the optical axis.

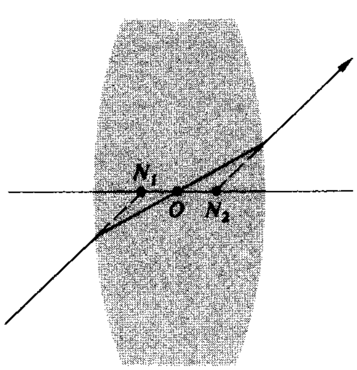


Figure 6.2 Nodal points.

Nodal points N_1 and N_2: the interception of the incident and emerged rays which pass the optical center with optical axis.

Cardinal Points: the two focal, two principal and two nodal points.

Thick Lens Formula

Single Lens

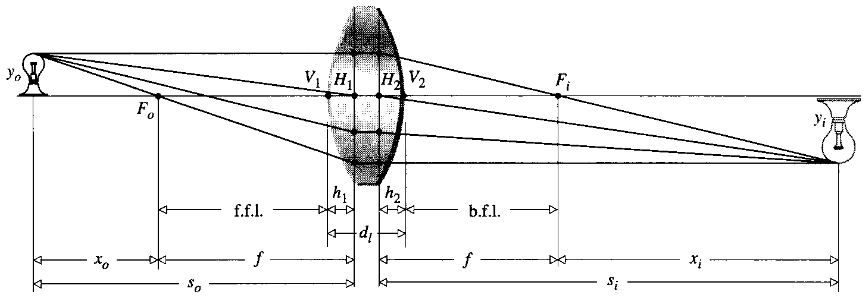


Figure 6.4 Thick-lens geometry.

If consider the thick lens as the combination of two spherical refracting surface separated by a distance d_l, the result is

$$\frac{1}{s_o}+\frac{1}{s_i}=\frac{1}{f}$$

$$\frac{1}{f}=(n_l-1)\left[\frac{1}{R_1}-\frac{1}{R_2}+\frac{(n_l-1)d_l}{n_l R_1 R_2}\right]$$

Note that s_o, s_i and f are measured from the first and second principal planes. Also the distance of the principal points and the vertices $\overline{V_1H_1}=h_1$ and $\overline{V_2H_2}=h_2$ are

$$h_1=-\frac{f(n_l-1)d_l}{R_2 n_l}$$

$$h_2=-\frac{f(n_l-1)d_l}{R_1 n_l}$$

which are positive when the principal points lie to the right of their respective vertices.

Double Lens

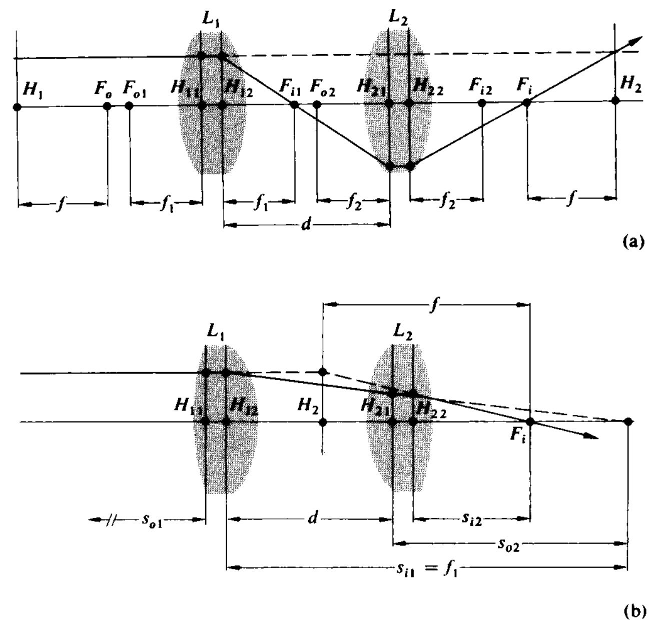


Figure 6.5 A compound thick lens.

The focus becomes

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{d}{f_1 f_2}$$

The principle planes are

$$\overline{H_{11}H_1}=\frac{fd}{f_2}$$

$$\overline{H_{22}H_2}=\frac{fd}{f_1}$$

Analytical Ray Tracing(6.2)

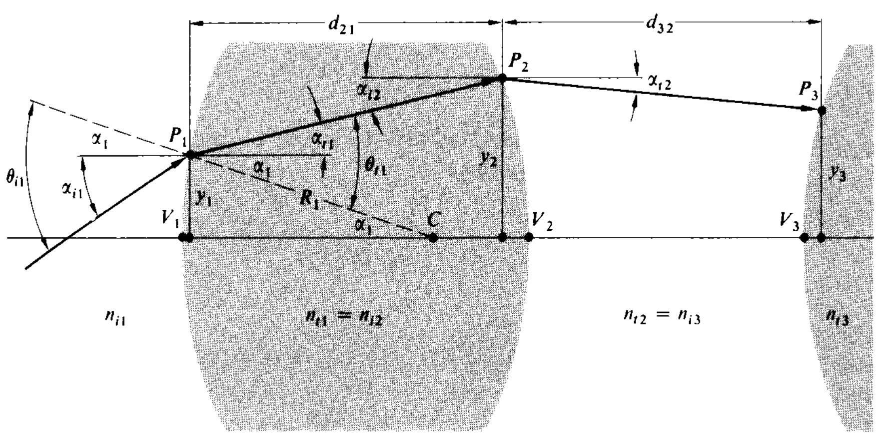


Figure 6.7 Ray geometry.

At the First Interface

From Snell's Law with paraxial approximation,

$$n_{i1}\sin\theta_{i1}=n_{t1}\sin\theta_{t1} \Rightarrow n_{i1}\theta_{i1}\approx n_{t1}\theta_{t1}$$

Since, $\theta_{i1} = \alpha_{i1} + \alpha_1$, $\theta_{t1} = \alpha_{t1} + \alpha_1$ and $\alpha_1 \approx \frac{y_1}{R_1}$, we have

$$n_{i1}(\alpha_{i1} + \frac{y_1}{R_1}) = n_{t1}(\alpha_{t1} + \frac{y_1}{R_1}) \Rightarrow n_{t1}\alpha_{t1} = n_{i1}\alpha_{i1} - \left(\frac{n_{t1} - n_{i1}}{R_1} \right) y_1$$

Since $D_1 = \frac{n_{t1} - n_{i1}}{R_1}$, we have

$$n_{t1}\alpha_{t1} = n_{i1}\alpha_{i1} - D_1 y_1$$

This is called the **refraction equation** pertaining to the first interface.

From the First Interface to the Second

$$y_2 = y_1 + d_{21}\tan\alpha_{t1} \approx y_1 + d_{21}\alpha_{t1}$$

This is known as the transfer equation.

At the Second Interface

$$n_{t2}\alpha_{t2} = n_{i2}\alpha_{i2} - D_2 y_2$$

where

$$D_2 = \frac{n_{t2} - n_{i2}}{R_2}, \quad n_{i2} = n_{t1}, \quad \alpha_{i2} = \alpha_{t1}$$

Matrix Method

At the first interface

$$\begin{bmatrix} n_{t1}\alpha_{t1} \\ y_{t1} \end{bmatrix} = \begin{bmatrix} 1 & -D_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} n_{i1}\alpha_{i1} \\ y_{i1} \end{bmatrix}$$

Note that in this case $y_{t1} = y_{i1}$

let

$$r_{t1}=\begin{bmatrix} n_{t1}\alpha_{t1} \\ y_{t1} \end{bmatrix},\ r_{i1}=\begin{bmatrix} n_{i1}\alpha_{i1} \\ y_{i1} \end{bmatrix},\ R_1=\begin{bmatrix} 1 & -D_1 \\ 0 & 1 \end{bmatrix}$$

then

$$r_{t1}=R_1 r_{i1}$$

R_1 Is called the **refraction matrix**.

Similarly, we can define a **transfer matrix** F_{21} to relate the ray from the first interface to the second inside the lens. We have

$$r_{i2}=F_{21} r_{t1}$$

where

$$r_{i2}=\begin{bmatrix} n_{i2}\alpha_{i2} \\ y_{i2} \end{bmatrix},\quad F_{21}=\begin{bmatrix} 1 & 0 \\ \dfrac{d_{21}}{n_{t1}} & 1 \end{bmatrix}$$

At the second interface, we have

$$r_{t2}=R_2 r_{i2}$$

where

$$r_{t2}=\begin{bmatrix} n_{t2}\alpha_{t2} \\ y_{t2} \end{bmatrix},\quad R_2=\begin{bmatrix} 1 & -D_2 \\ 0 & 1 \end{bmatrix}\ \backslash$$

To sum up, we have

$$r_{t2}=R_2 F_{21} R_1 r_{i1}$$

Let $A=R_2 F_{21} R_1$, then A is called the system matrix of the optical system.

Substitute $n_{t1}=n_{i2}=n_l,\ d_{21}=d_l$, then

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 - \frac{D_2 d_l}{n_l} & -D_1 - D_2 + \frac{D_1 D_2 d_l}{n_l} \\ \frac{d_l}{n_l} & 1 - \frac{D_1 d_l}{n_l} \end{bmatrix}$$

Note that

$$-a_{12} = D_1 + D_2 - D_1 D_2 \frac{d_l}{n_l} = -\frac{1}{f}$$

according to the thick lens formula.

Image Formation

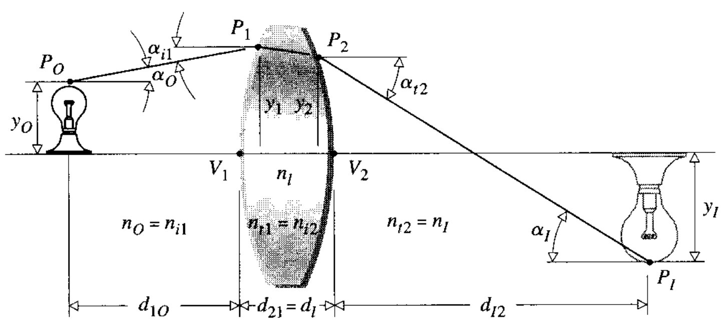


Figure 6.8 Image geometry.

Suppose the object point is located at Point P_O and the image P_I, then

$$r_I = F_{I2} A_{21} F_{1O} r_O$$

which describe how the rays travel from P_O to P_I where F_{1O} is the transfer matrix from P_O to the lens and F_{I2} from the lens to P_I. However, we do not know F_{I2} yet since we are not sure where the image will form. By expanding the compete system matrix, it is possible to find the location of the image such that all rays from P_O meet at P_I.

Aberrations

Chromatic aberrations: due to the fact that refraction index is a function of frequency

monochromatic aberrations: spherical aberration, coma, astigmatism, Petzval field curvature and distortion.

Spherical Aberration

Since

$$\sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \cdots$$

if instead of $\sin\phi \approx \phi$, we keep the third order term, that is $\sin\phi \approx \phi - \frac{\phi^3}{3!}$.

This is called third order approximation. Apply this to the origin derivation of the formula of the refraction of a spherical interface, we have a more accurate formula as follow

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} + h^2\left[\frac{n_1}{2s_o}\left(\frac{1}{R} + \frac{1}{s_o}\right)^2 + \frac{n_2}{2s_i}\left(\frac{1}{R} - \frac{1}{s_i}\right)^2\right]$$

Comparing to the original formula

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R},$$

it is obviously that the h^2 is the correction term. The result is a dependency of the image location on h, the distance of the ray t the interface to the optical axis.

Table 9A VALUES OF $\sin\theta$ AND ITS FIRST THREE EXPANSION TERMS

	$\sin\theta$	θ	$\frac{\theta^3}{3!}$	$\frac{\theta^5}{5!}$
10°	0.1736482	0.1745329	0.0008861	0.0000135
20°	0.3420201	0.3490658	0.0070888	0.0000432
30°	0.5000000	0.5235988	0.0239246	0.0003280
40°	0.6427876	0.6981316	0.0567088	0.0013829

Longitudinal Spherical aberration ($L \cdot SA$): the distance between the axial intersection of a ray and the paraxial focus.
Positive SA: the marginal rays intersect the central axis before the paraxial focus. Usually, when the lens is convergent.
Negative SA: the marginal rays intersect the central axis after the paraxial focus. Usually, when the lens is divergent.

Traverse(lateral) Spherical aberration ($T \cdot SA$): the height above the central axis where the ray intercepts the paraxial focal plane.

Circle of least confusion ($\sum_{LC}$): for an object point in infinity, the plane where the image has the smallest diameter.

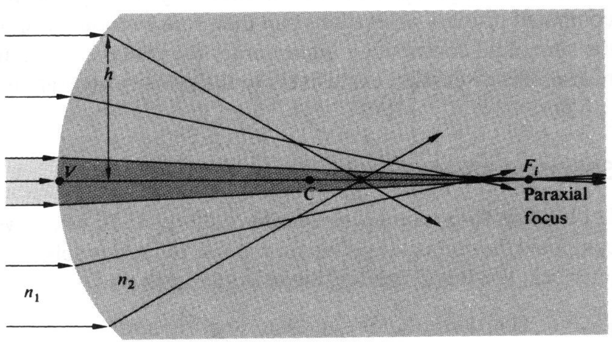


Figure 6.13 Spherical aberration resulting from refraction at a single interface.

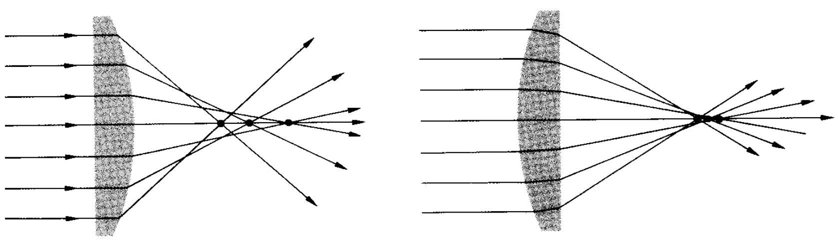


Figure 6.16 SA for a planar-convex lens.

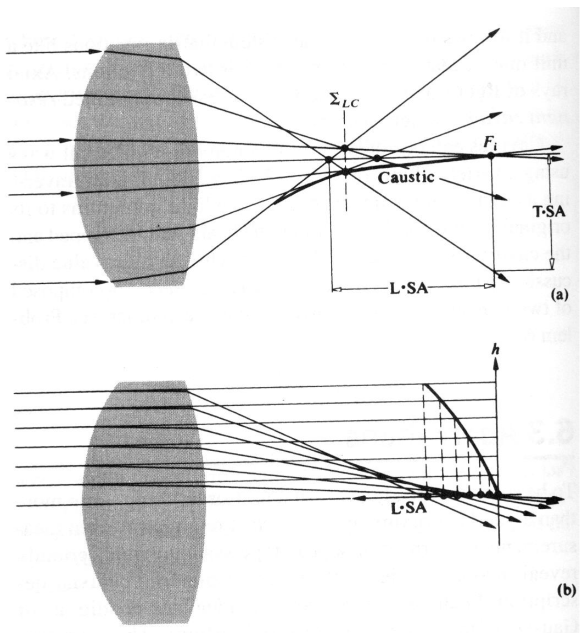


Figure 6.14 Spherical aberration for a lens. The envelope of the refracted rays is called a caustic. The intersection of the marginal rays and the caustic locates Σ_{LC}.

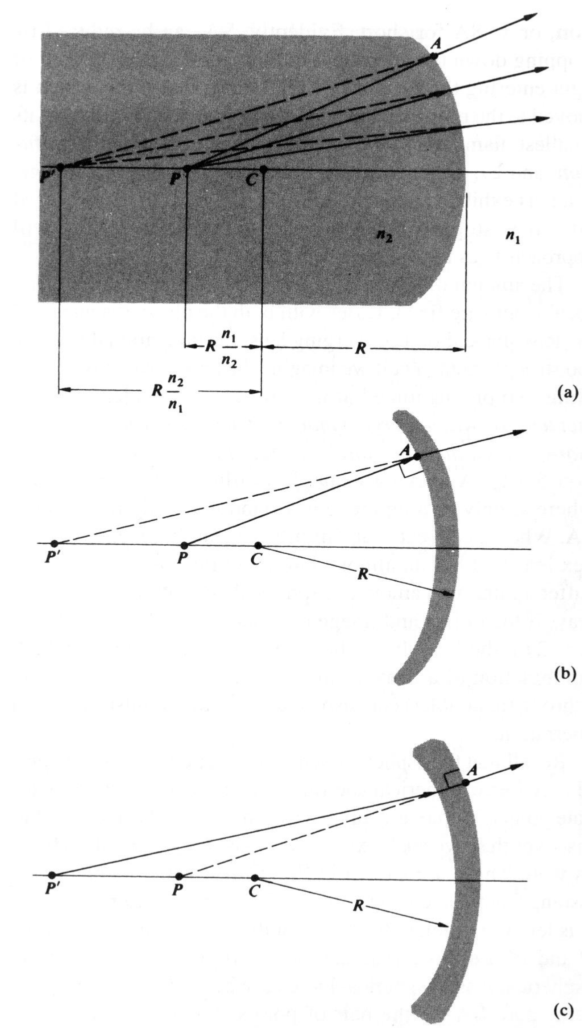


Figure 6.17 Corresponding axial points for which SA is zero.

Reducing SA:

1. Reducing the aperture such that the decreases the number off-axis rays. However, also reduces the amount of light entering the system.
2. Carefully choosing the lens.
3. Choose the right locations of the object and image;
   a. Let $s_o=f=s_i$. If the lens is symmetry, then the deviation of the ray is minimum.
   b. Find the no SA points.
4. Smallest spherical aberration occurs when $q=+0.7$, where

$$q = \frac{r_2 + r_1}{r_2 - r_1}.\ q \text{ Is called shape factor.}$$

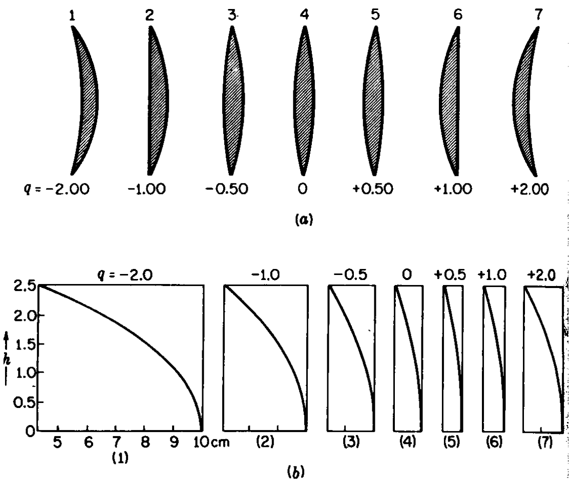


FIGURE 9F
(*a*) Lenses of different shapes but with the same power or focal length. The difference is one of bending. (*b*) Focal length versus ray height *h* for these lenses.

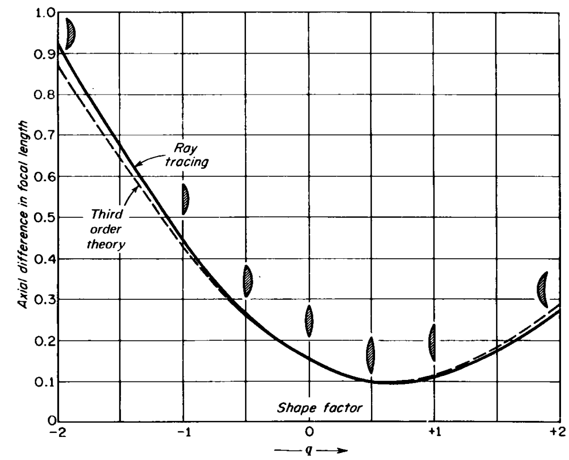


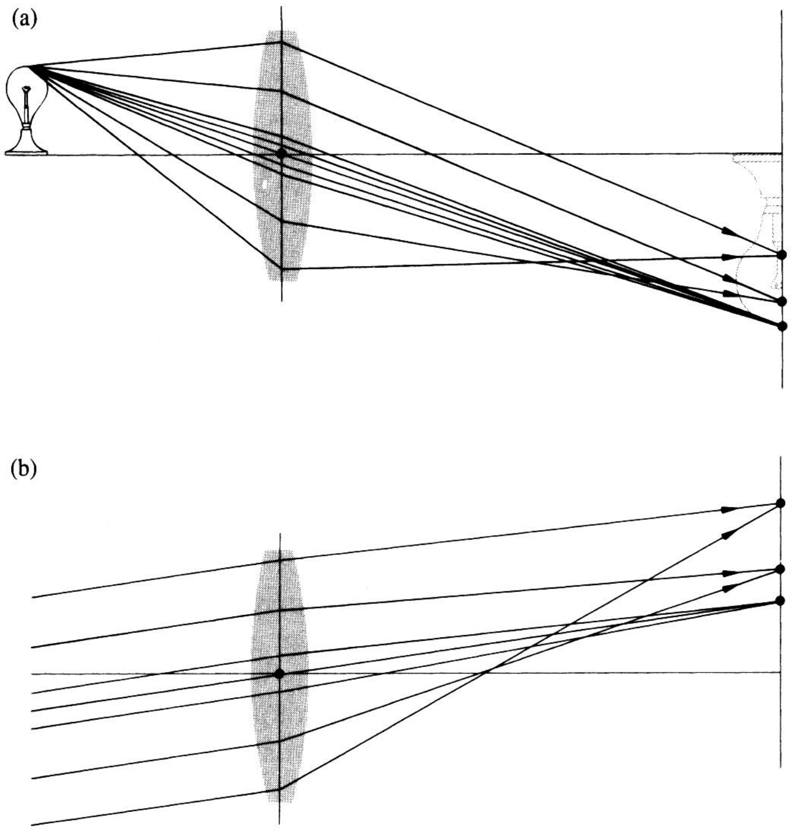
FIGURE 9G
A graph of the spherical aberration for lenses of different shape but the same focal length. For the lenses shown $h = 1$ cm, $f = +10$ cm, $d = 2$ cm, and $n' = 1.51700$.

Table 9B SPHERICAL ABERRATION OF LENSES HAVING THE SAME FOCAL LENGTH BUT DIFFERENT SHAPES q
Lens thickness = 1 cm $f = 10$ cm $n = 1.5000$ $h = 1$ cm

Shape of lens	r_1	r_2	q	Ray tracing	Third-order theory
Concavo-convex	−10.000	− 3.333	−2.00	0.92	0.88
Plano-convex	∞	− 5.000	−1.00	0.45	0.43
Double convex	20.000	− 6.666	−0.50	0.26	0.26
Equiconvex	10.000	−10.000	0	0.15	0.15
Double convex	6.666	−20.000	+0.50	0.10	0.10
Plano-convex	5.000	∞	+1.00	0.11	0.11
Concavo-convex	3.333	10.000	+2.00	0.27	0.29

Coma (comatic aberration)

In the absence of SA, the light coming off an off-axis object point does not focus at one single point on the image plane is called **coma**.



1. Negative coma: the focus of marginal rays is closer to the central axis than principle rays
2. positive coma: the focus of marginal rays is further to the central axis than principle rays
3. Coma will causes interference.
4. Coma is dependent on the shape of the lens.

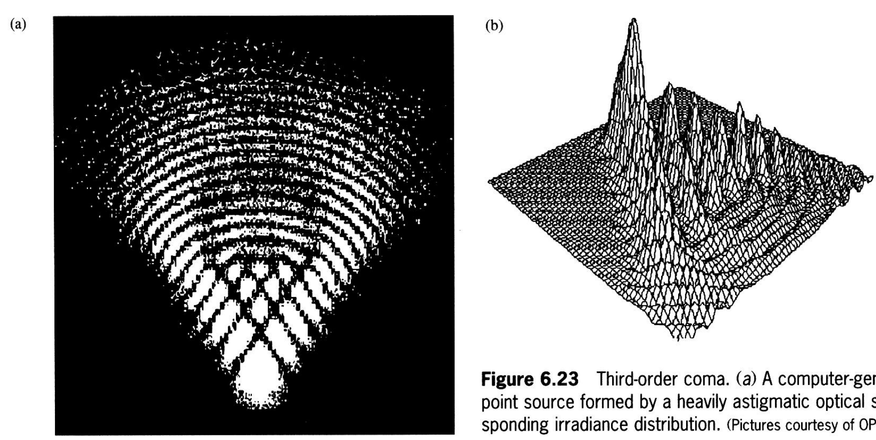


Figure 6.23 Third-order coma. (*a*) A computer-generated diagram of the image of a point source formed by a heavily astigmatic optical system. (*b*) A plot of the corresponding irradiance distribution. (Pictures courtesy of OPAL Group, St. Petersburg, Russia.)

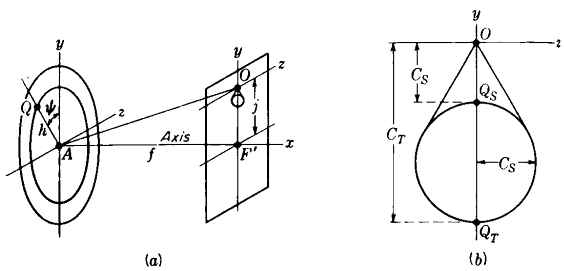


FIGURE 9K
Geometry of coma, showing the relative magnitudes of sagittal and tangential magnifications.

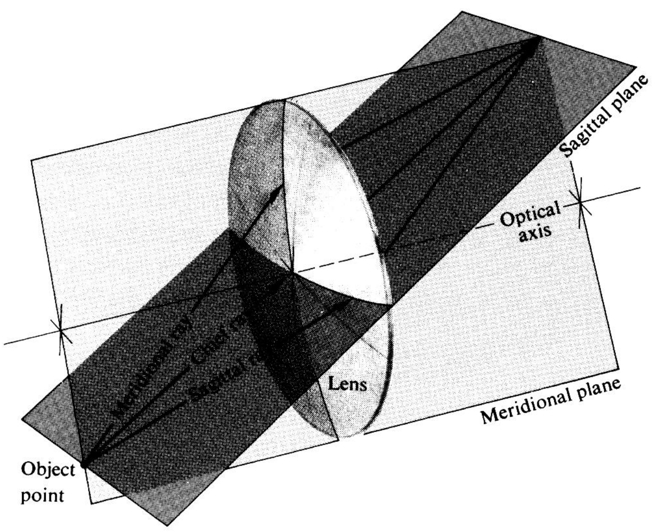
Table 9D COMPARISON OF COMA AND SPHERICAL ABERRATION FOR LENSES OF THE SAME FOCAL LENGTH BUT DIFFERENT SHAPE FACTOR

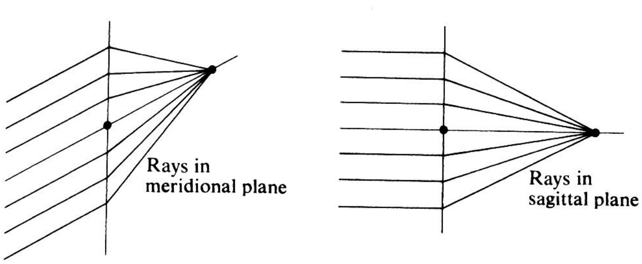
$h = 1.0$ cm $f = +10.0$ cm $y = 2.0$ cm
$n = 1.5000$

Shape of lens	Shape factor	Coma, cm	Spherical aberration, cm
Concavo-convex	−2.0	−0.0420	+0.88
Plano-convex	−1.0	−0.0270	+0.43
Double convex	−0.5	−0.0195	+0.26
Equiconvex	0	−0.0120	+0.15
Double convex	+0.5	−0.0045	+0.10
Plano-convex	+1.0	+0.0030	+0.11
Concavo-convex	+2.0	+0.0180	+0.29

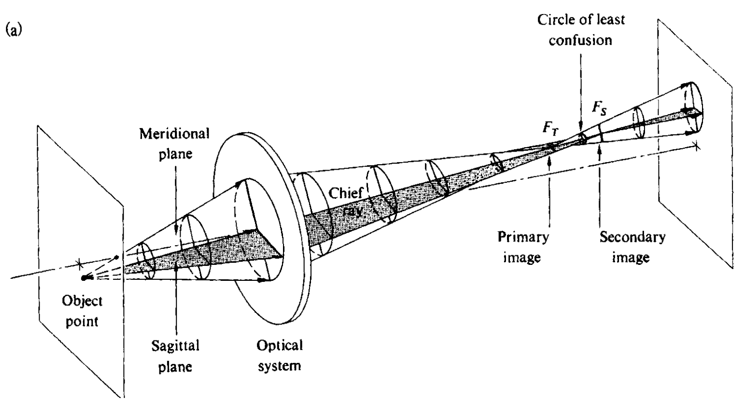
Shape factor q=0.8 for no coma is near the shape factor q=0.714 for minimum spherical aberration.

Astigmatism





1. Caused by the difference in ray configuration in tangential (meridional) and sagittal planes of an off-axis object point.
2. The rays from the off-axis object point focus to two lines called tangential and sagittal focus, F_T and F_S.
3. Astigmatism is approximately proportional to the focal length and is very little improved by changing



the shape of the lens.

4. Loci of the tangential and sagittal images approximately form two paraboloids.

5. Astigmatism can be reduced by proper spacing of the lens elements or by the

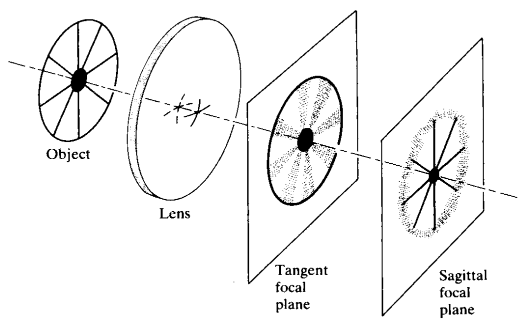
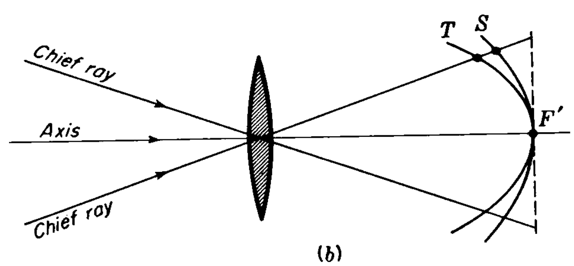


Figure 6.28 Images in the tangent and sagittal focal planes.

proper location of a stop.

6. When astigmatism is completed removed, the two loci of the tangential and sagittal images form into one paraboloidal surface, called Petzval surface.



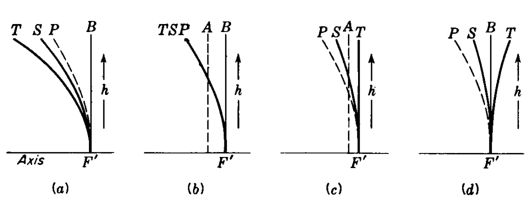
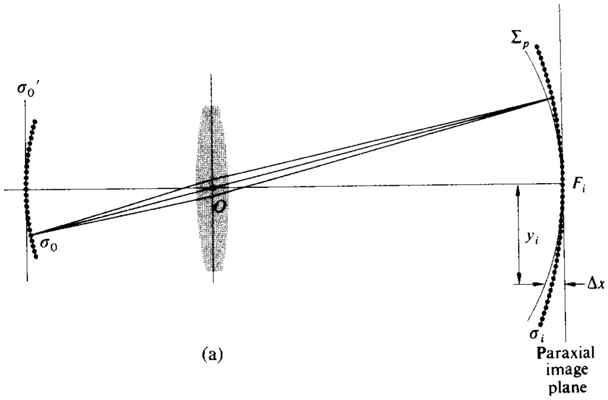


FIGURE 9R
Diagrams showing the astigmatic surfaces T and S in relation to the fixed Petzval surface P as the spacing between lenses (or between lens and stop) is changed.

Field Curvature



When no SA, coma and astigmatism exist, a planar object normal to the axis will imaged into a curved surface instead of a plane in the paraxial region. This aberration is known as Petzval field curvature.

Let Δx be the distance of an image point at height h_i on the Petzval surface form the paraxial image plane. Then,

$$\Delta x = \frac{y_i^2}{2} \sum_{j=1}^{m} \frac{1}{n_j f_j}$$

for an m-lens combination. Note that the spacings of the lenses has no effect on Δx. Suppose $m=2$,

$$\Delta x = 0 \ \Rightarrow \ \frac{1}{n_1 f_1} + \frac{1}{n_2 f_2} = 0 \ \Rightarrow \ n_1 f_1 + n_2 f_2 = 0.$$

This is called Petzval condition.

Example: $f_1 = -f_2$, $n_1 = n_2$.

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} \ \Rightarrow \ f = \frac{f_1^2}{d}$$

Distortion

1. Distortion is caused by the fact that the transverse magnification, M_T, is a function of the off-axis image distance. In other word, distortion arises because different areas of the lens have different local lengths.
2. Positive or pincushion distortion: M_T increases with the axial distance.
3. Negative or barrel distortion: M_T decreases with the axial distance.

4. Effect of stop
    a. Distortion is very small for a thin positive lens if the aperture is located at the lens.
    b. Stop after the lens causes positive distortion.
    c. Stop before the lens causes negative distortion.
    d. If two lens are perfectly symmetrical with a stop at the middle, the distortion can be cancelled.

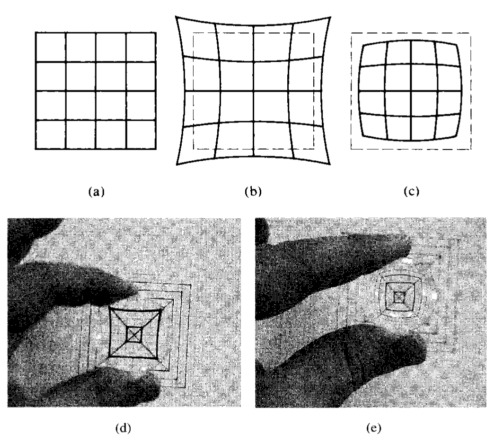


Figure 6.33 (*a*) Undistorted object. (*b*) When the magnification on the optical axis is less than the off-axis magnification, pincushion distortion results. (*c*) When it is greater on axis than off, barrel distortion results. (*d*) Pincushion distortion in a single thin lens. (*e*) Barrel distortion in a single thin lens. (Photos by E.H.)

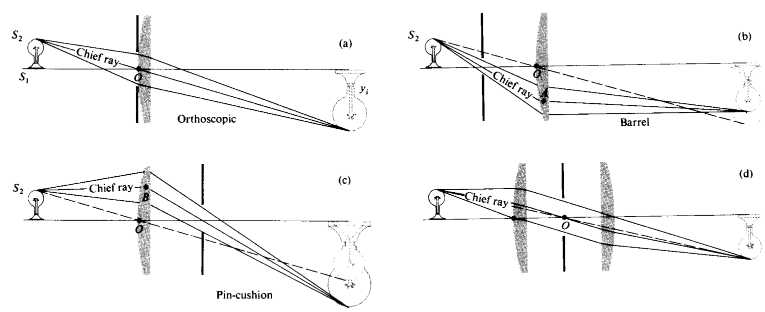


Figure 6.34 The effect of stop location on distortion.

Chromatic aberrations

1. Due to the dependency of the refraction index on frequency. Lights with different wavelengths have different focuses.
2. A·CA**:** axial chromatic aberration.
3. L·CA**:** lateral chromatic aberration.

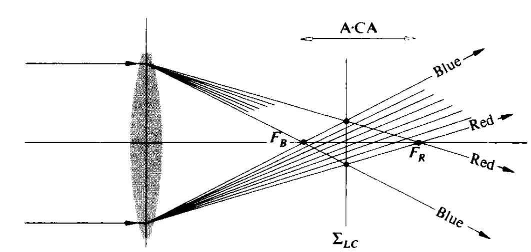


Figure 6.36 Axial chromatic aberration.

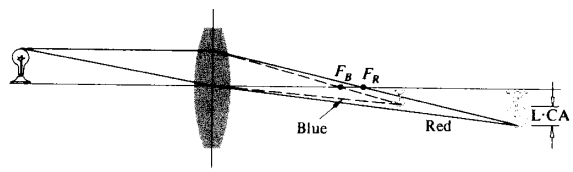


Figure 6.37 Lateral chromatic aberration.

Thin Achromatic Doublets

Purpose: to bring the focus of the red and blue lights together by a combination of two thin lens separated by a distance d.

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{d}{f_1 f_2}$$

Let

$$\frac{1}{f_1}=(n_1-1)(\frac{1}{R_{11}}-\frac{1}{R_{12}})=(n_1-1)\rho_1,\ \frac{1}{f_2}=(n_2-1)(\frac{1}{R_{21}}-\frac{1}{R_{22}})=(n_2-1)\rho_2,$$

then

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{d}{f_1 f_2}=(n_1-1)\rho_1+(n_2-1)\rho_2-d(n_1-1)\rho_1(n_2-1)\rho_2.$$

Let the focus of red light be f_R and blue light f_B. What we want is $\frac{1}{f_R}=\frac{1}{f_B}$.

This leads to

$$(n_{1R}-1)\rho_1+(n_{2R}-1)\rho_2-d(n_{1R}-1)\rho_1(n_{2R}-1)\rho_2=(n_{1B}-1)\rho_1+(n_{2B}-1)\rho_2-d(n_{1B}-1)\rho_1(n_{2B}-1)\rho_2$$

Case 1: Select $d=0$, we have

$$\frac{\rho_1}{\rho_2} = -\frac{n_{2B} - n_{2R}}{n_{1B} - n_{1R}}.$$

Let the focus of yellow light be f_Y, then

$$\frac{1}{f_{1Y}} = (n_{1Y} - 1)\rho_1, \ \frac{1}{f_{2Y}} = (n_{2Y} - 1)\rho_2 \ \Rightarrow \ \frac{\rho_1}{\rho_2} = \frac{(n_{2Y} - 1)f_{2Y}}{(n_{1Y} - 1)f_{1Y}}.$$

Therefore,

$$\frac{f_{2Y}}{f_{1Y}} = -\frac{(n_{2B} - n_{2R})/(n_{2Y} - 1)}{(n_{1B} - n_{1R})/(n_{1Y} - 1)}.$$

Definition:

1. Dispersive power: $\frac{n_B - n_R}{n_Y - 1}$.
2. Dispersive index, or V-number, or Abbe number: $V = \frac{n_Y - 1}{n_B - n_R}$.

Thus,

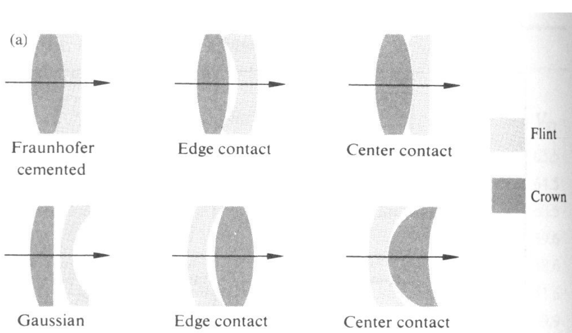
$$\frac{f_{2Y}}{f_{1Y}} = -\frac{V_1}{V_2} \Rightarrow f_{1Y}V_1 + f_{2Y}V_2 = 0$$

Case 2: Select $n_1 = n_2$, then

$$d = \frac{1}{n_B + n_R - 2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) = \frac{(f_{1Y} + f_{2Y})(n_Y - 1)}{n_B + n_R - 2}.$$

If $n_Y = \frac{n_B + n_R}{2}$, we have

$$d = \frac{f_{1Y} + f_{2Y}}{2}$$



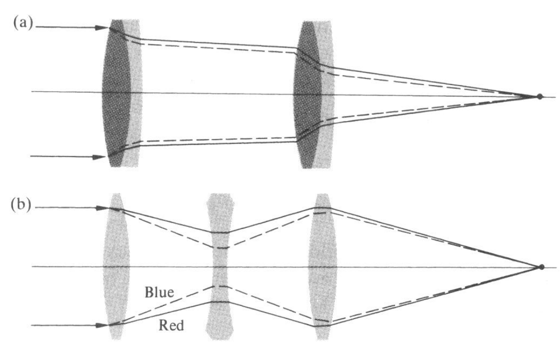


Figure 6.41 Achromatized lenses.

GRIN (GRadient INdex) Systems

1. Homogeneous lenses apply the difference between its refraction index and that of the surroundings medium, and the curvature of its interfaces to reconfigures a wavefront or the direction of the rays.
2. Same effect can be achieve by changing the refraction index profile of the lenses while keeping the lenses flat.
3. Radial-GRIN: refraction index varies axially.

Exampe:

If a parallel light is to focus at a point F as in the figure, all optical path must equal. For a ray located at distance r from the center,

$$n_{\max} d + f = (OPL)_0 = (OPL)_r \approx n(r) d + \overline{AB} + f \Rightarrow n_{\max} d \approx n(r) d + \overline{AB}.$$

Since $\overline{AF} \approx \sqrt{r^2 + f^2}$ and $\overline{AB} = \overline{AF} - f$,

$$n(r) = n_{\max} - \frac{\sqrt{r^2 + f^2} - f}{d}$$

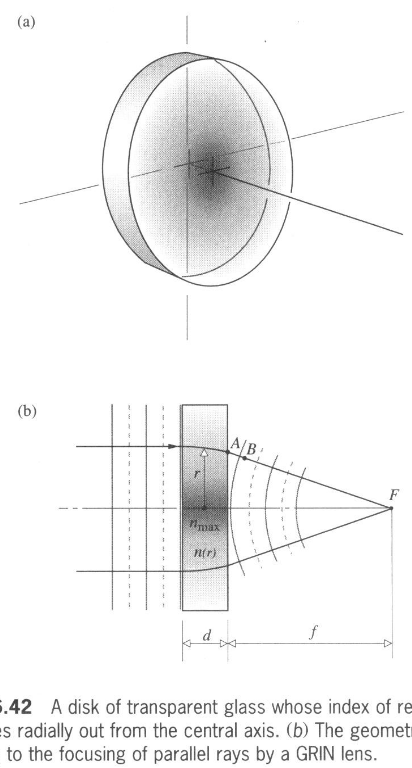


Figure 6.42 A disk of transparent glass whose index of refraction decreases radially out from the central axis. (*b*) The geometry corresponding to the focusing of parallel rays by a GRIN lens.

Typical index profile:

$$n(r) = n_{max}(1 - \frac{ar^2}{2}).$$

- Light propagates sinusoidally. The period in space: $\frac{2\pi}{\sqrt{a}}$
- By change the object distance or the length of the lens, an erect, real, magnified image can be formed.
- Definition:
    1. Gradient constant: $\sqrt{a}$.
    2. Pitch: thickness of the lens in term of the period.

Axial-GRIN

1. Refraction index varies axially instead of radially.
2. Also combined with curve surfaces.
3. Can correct spherical aberrations

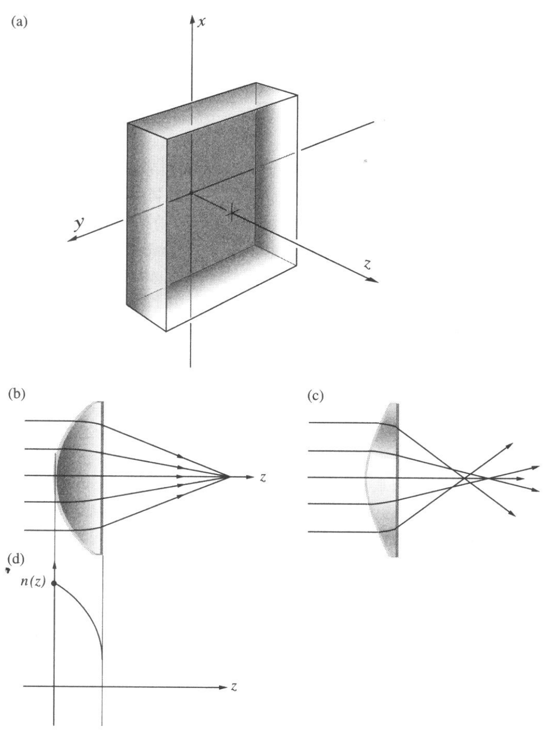


Figure 6.45 (*a*) A slab of axial-GRIN material for which the index of refraction is *n*(*z*). (*b*) An axial-GRIN lens for which there is no spherical aberration. (*c*) An ordinary lens having SA. (*d*) The index profile.

Optical sine theorem

$$n_o y_o \sin\alpha_o = n_i y_i \sin\alpha_i$$

where n_o, y_o, α_o, and n_i, y_i, α_i are the index, height and slope angle of the ray in object and image space, respectively.

If coma is to be zero,

$$M_T = \frac{y_i}{y_o} = \text{constant}$$

for all α_o. For paraxial rays: $\sin\alpha_o = \alpha_{op}$, $\sin\alpha_i = \alpha_{ip}$, therefore

$$\frac{\sin\alpha_o}{\sin\alpha_i} = \frac{\alpha_{op}}{\alpha_{ip}} = \text{constant}$$

which is known as the Sine Condition.